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Journal of Sound and Vibration 302 (2007) 1030–1036
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Vibration of plates with arbitrary shapes of cutouts

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Received 22 May 2006; received in revised form 30 August 2006; accepted 7 January 2007
Available online 26 February 2007

Abstract

Amalgamation of a subparametric triangular plate bending element with first-order shear deformation has occurred for the first time with an approach that maintains uniform mesh sizes and shapes even while dealing with cutouts of arbitrary shapes. This is a distinct improvement over the existing practices of cutout modeling. Further the formulation being based on the subparametric element has the advantage of achieving matching modes, which enables the model to deal problems of very thin plates without even going for reduced integration. Numerical examples on free vibration of plates with cutouts have been analysed and the results presented together with those available in published literature.

1. Introduction

Cutouts of regular geometric shapes are designed for various purposes [1] in diverse engineering fields. Sometimes a cutout is not a part of initial design [2]. It happens because of damage of the structural elements during their service life. In areas of biomechanics the analysis of cutouts of irregular shapes are also encountered [3]. This brings the need for suggesting methods to deal arbitrary shaped cutouts. Though the present concern is free vibration of plates, the review covers static and buckling problems of plates and shells as well to exhibit the existing trend of cutout modeling.

No doubt, some semi-analytical approaches for dealing problems on cutouts of shapes other than rectangular and of arbitrary ones exists [2,4,5] but any general purpose finite element method dealing such problems is yet to be found. The approaches adopted till date cannot be called general purpose because the discretization shape, number and overall meshing pattern become unique for each case. The number of elements not only depends on the size of the cutout but also on its shape and even on its location in the structure. Moreover, as the element shapes and sizes throughout the structure do not remain uniform, the use of identical plate elements for computation of element matrices is not possible. Thus these methods increase the cost of generation of global matrices because of repetitive computation. To be clearer, the case of a rectangular cutout be considered. The mesh divisions towards the cutout boundary become finer [6] thereby causing an increase in the total number of elements, which would have been much less, if there had been no cutout. Some of the publications, dealing with cutouts, even make no mention of the method they adopt for

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.01.003

Nomenclature

a, b	length and width of plate
a', b'	length and width of cutout
E	Young's modulus
υ	Poisson's ratio
ρ	mass density of plate
ϖ	non-dimensional fundamental frequency $(=\omega b^2[12\rho(1-\upsilon^2)/Eh^2]^{1/2})$

modeling the cutouts [7–13]. Some quadrilaterals of different sizes around the cutout as cited by some researchers [14,15] have been used by ABAQUS. This is also the method of handling the cases of circular and elliptical cutouts [16,17]. In all these cases the mesh lines have been aligned around the cutout boundary. This method of aligning the mesh lines around the cutout boundary not only increases the total degrees of freedom (dof) but also raises a question about the uniformity in the level of precision in the calculation of the stiffness and the mass matrices. This is because, for example the investigators [16,17] have integrated the element matrices by reduced integration to get rid of shear locking. Though this may give some degree of accuracy to the element matrices of the rectangular elements (away from the cutout) the characteristic matrices of the quadrilateral elements (near the cutouts) being fractions of polynomials, would need more Gauss points to achieve the same degree of accuracy [18]. Moreover, the other deficiencies of these elements not being compact are also inherent in them. These problems become more pronounced if an arbitrarily shaped perforation is encountered, in case of which a good number of quadrilaterals and triangles will have to surround the cutout. All these shortcomings can be eliminated if the plate can be discretized into a number of square and compact elements, notwithstanding the shape and size of the hole. This can be achieved if the cutout is also divided along with the plate so that portions of the cutout can now be treated as elements within the plate element. Then the element matrices of the holed plate can be obtained by deducting the corresponding matrices of the cutout from those of the solid plate element. A little bit of elaboration of this technique seems to be the point in order at this stage. Fig. 1 shows a rectangular plate containing cutouts of arbitrary shapes. The plate is discretized into 18 numbers of triangular elements of uniform shape and size. Elements numbered as 9, 10, 11, 16, 17 and 18 contain portions of the cutout. If these portions are considered as elements, the displacement at any point within such an element can be interpolated from the nodal values of that element. The nodal displacement values of any particular cutout element can in turn be interpolated from the nodal displacements of the corresponding triangular plate element within which the cutout portion is contained. As a result the displacement at any point of the cutout element gets expressed finally in terms of the nodal displacements of

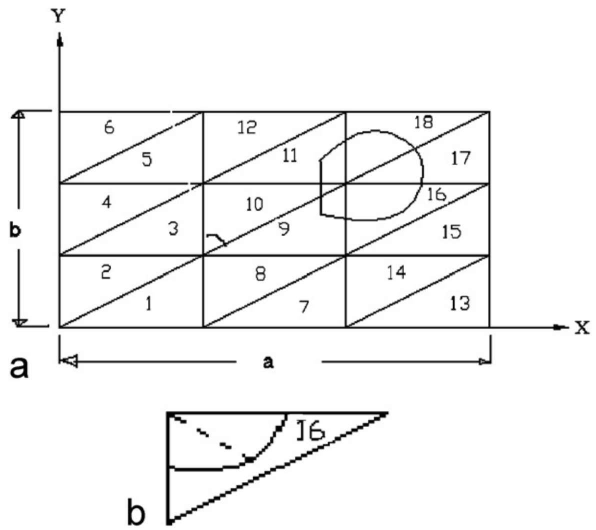

Fig. 1. (a) Uniform mesh layout in a rectangular plate with cutouts of arbitrary shapes, and (b) cutout in element 16 modeled as two triangular cutouts.

the virgin plate element. The entire process leading to the evaluation of the element matrices of the cutout element is clearly explained in the following section. Though such a technique appears in the name of negative stiffness method [19], the capability of the element used there to implement the 'method of deduction' in the case of very thin plates is doubtful, because the authors have used full integration scheme. Even if someone adopts the reduced integration technique to solve the cases of thin plates, the occurrences of spurious zero energy modes cannot be averted. Thus the resulting element matrices obtained by using the 'method of deduction' may produce non-positive pivots even in the case of moderately thin plates. Thus though the 'method of deduction' is a convenient tool to deal arbitrary shapes of cutouts it becomes ineffective in the case of thin plates because of these reasons.

This discussion makes it clear that an ideal model for cutouts would be the one that can use the 'method of deduction' both in thick and very thin plates. This is possible through an element that ensures the existence of matching modes for the rotational terms [20], which is the basis of the present investigation.

In this paper, the plate and the cutout are analysed using a 13-noded subparametric triangular element. The stiffness and mass matrices of the cutout are calculated using the standard principles of finite element analysis. Then these matrices are transformed to all the nodes of the plate element by considering the portion of the cutout within the plate element.

2. Finite element formulation

A new C^0 plate element, which includes transverse shear deformation, is formulated. The polynomials adopted for the deformation fields are complete. The stress normal to the middle surface is assumed to be negligible. Material is isotropic and linearly elastic. As the polynomials are of higher order, faster rate of convergence can be achieved. Three dof are considered here i.e. transverse displacement (w), rotation about y-axis (α) and rotation about x-axis (β). To get the matching modes as discussed by Koziey and Mirza [20] the degree of polynomial for approximation of w is cubic whereas those for the rotations are assumed to be quadratic. To construct the cubic approximation for w the displacement at the nodes 1, 2, 3, 4, 6, 7, 9, 10, 12 and 13 are used. The quadratic approximation of rotation α and β are achieved using the rotation at the nodes 1, 2, 3, 5, 8 and 11 as shown in Fig. 2. The sign convention for the dof has been followed from standard text [18]. The strain terms as well as the elasticity matrix $[D]$ are followed from Mindilin plate element of the same text.

The location of any point within the element in the global coordinate system is determined by the coordinates of the nodal points (x_i, y_i) as

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \sum_{i=1}^{13} N_i \begin{Bmatrix} x_i \\ y_i \end{Bmatrix}. \tag{1}$$

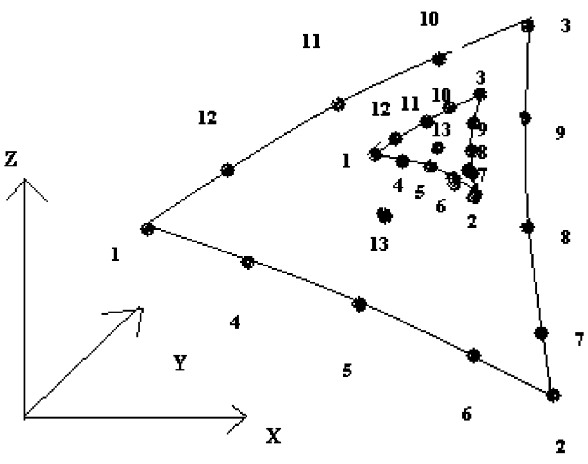

Fig. 2. Subparametric plate element in the x–y plane with discretized cutout inside.

The global displacement w and the rotations can be expressed in terms of nodal displacements and rotations as

$$w = \sum_{i=1}^{13} \bar{N}_i w_i, \tag{2}$$

$$\left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} = \sum_{i=1}^{13} N_i \left\{ \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right\}, \tag{3}$$

where $\bar{N}_i$ and N_i are the cubic and quadratic shape functions respectively, which are expressed in non-dimensional area coordinates.

N_4, N_6, N_7, N_9, N_{10}, N_{12}, N_{13} and $\bar{N}_5, \bar{N}_8, \bar{N}_{11}$ are all zeros. The remaining shape functions can be found from any standard text.

The stiffness matrix of the solid plate element

$$[K_{se}] = \int_A [B]^{\mathrm{T}}[D][B]h \, \mathrm{d}A. \tag{4}$$

The mass matrix of the solid plate element

$$[M_{se}] = \int_A [N]^{\mathrm{T}}[P][N]h \, \mathrm{d}A. \tag{5}$$

The inertia matrix $[P]$ is obtained as follows:

$$[P] = [I]_{3\times 3} \left[\rho h, \frac{\rho h^3}{12}, \frac{\rho h^3}{12} \right]^{\mathrm{T}}, \tag{6}$$

where h is the thickness of the plate and A, the area of the triangular element.

$[B]$ and $[N]$ are the strain–displacement and the shape function matrices, respectively.

The displacement at any point within the cutout element can be found as

$$w_c = \sum_{i=1}^{13} \bar{N}_{ci} w_{ci} = \sum_{i=1}^{13} \bar{N}_{ci} \sum_{i=1}^{13} \bar{N}_i w_i. \tag{7}$$

The subscript c stands for cutout.

Similarly α_c and β_c can be found out in terms of the nodal rotations of the plate element. Thus

$$\sum_{i=1}^{13} \{\delta_c\}_i = [T] \sum_{r=1}^{13} \{\delta\}_r, \tag{8}$$

where $\{\delta_c\}$ and $\{\delta\}$ are the displacement vectors for the cutout and plate respectively.

$$[T] = \sum_{i=1}^{13} \sum_{r=1}^{13} \begin{bmatrix} \overline{N_{ir}} & & 0 \\ & N_{ir} & \\ 0 & & N_{ir} \end{bmatrix}, \tag{9}$$

$$[K_{ce}] = [T]^{\mathrm{T}} \left[\overline{K_{ce}} \right] [T], \tag{10}$$

where

$$\left[\bar{K}_{ce} \right] = \int_A [B_c]^{\mathrm{T}}[D][B_c]h \, \mathrm{d}A. \tag{11}$$

The above integrands can be expressed as

$$\int_A \Phi \, \mathrm{d}A = \frac{1}{2} \sum_{i=1}^{ng} W_i J_i \Phi_i, \tag{12}$$

in which ϕ_i is the value of ϕ at a specific point in the triangle, W_i is the weight appropriate to this point, J_i is the determinant of the Jacobian evaluated at that point and ng is the number of sampling points used. It may be noted that weights are expressed directly here rather than as the product of one-dimensional weights [18].

All the above integrations have been carried out over the area of the triangle using 12 point Gauss quadrature formula to achieve the required degree of precision, which is equal to 6 here. The stiffness matrix of the perforated plate element is found by deducting the stiffness matrix of the hole from the stiffness matrix of the virgin plate element:

$$[K_e] = [K_{se}] - [K_{ce}]. \tag{13}$$

If the virgin element contains more than one cutout (element 10 in Fig. 1) or if a single cutout within the virgin element is to be subdivided into more than one triangle, as shown for element 16 in Fig. 1(b), Eq. (12) can easily be extended as

$$[K_e] = [K_{se}] - \sum_{i=1}^{N_c} [K_{ce}]_i, \tag{14}$$

where N_c indicates the number of the small cutouts.

Similar expressions can be written for the mass matrix of the perforated plate element. The resulting eigenvalue problem after constructing the global element matrices is solved by subspace iteration method [21].

3. Numerical examples

Numerical examples are solved for checking the accuracy of the model. The 1st problem considers fundamental frequency of a square plate of isotropic material ($\upsilon = 0.3$) having a central square cutout of various sizes for different side to thickness ratios. The purpose is to verify the capability of the model to handle problems in the range of thick to thin plates, without producing negative pivots while dealing with very thin plates. The results along with those of Reddy [22] are presented in Table 1. It is worth mentioning here that the problems solved by Tham et al. [19] involve plates having thickness to span ratio = 0.01 only, whereas in the present investigation the same ratio ranges from 0.2 to 0.001. The above mentioned cases have been solved by a 3×3 mesh (18 elements).

The 2nd problem is solved to show the ability of the model to deal problems of free vibration of plates having holes of curved sides. The element used to model the cutout being triangular fits well to curved boundaries. This problem considered earlier by Takahashi [23] and later by Bicos and Springer [12] computes the fundamental frequencies of clamped aluminum rectangular plate of varying circular cutout diameter. The authors [12] had used a 4-noded quadrilateral element with selective reduced integration. The results obtained by the present method have been compared with these two earlier observations in Table 2. For large cutouts the results match well with those of Takahashi's experiment [23]. For smaller diameters the difference is within 3%. The discretization of the plate exhibiting uniform mesh layout is shown in Fig. 3. All cases of this problem have been solved using a grid comprising of 18 squares (6×3).

Table 1
Non-dimensional fundamental frequencies (ϖ) of simply supported isotropic square plate with a central square cutout

a'/a	$a/h = 5$		$a/h = 10$		$a/h = 1000$	
	Present method	Ref. [22]	Present method	Ref. [22]	Present method	Ref. [22]
0.2	17.55	17.452	19.07	18.679	20.08	19.200
0.4	19.27	19.163	21.03	20.246	21.00	20.807
0.5	21.62	21.554	23.01	22.804	23.75	23.515
0.6	26.06	25.688	27.54	27.379	29.85	28.453
0.8	44.10	44.069	51.55	51.465	58.07	57.512

Table 2
Fundamental frequencies (Hz) of clamped aluminum rectangular plate of varying circular cutout radius r

r/b	Ref. [12]	Ref. [23]	Present method
0.000	148.77	148.77	151.00
0.032	148.77	148.77	153.01
0.068	148.78	150.00	154.00
0.100	148.79	151.23	155.02
0.132	150.00	154.91	157.00
0.168	154.91	159.84	162.10

$a = 0.504\,\text{m}$, $b = 0.254\,\text{m}$, $h = 0.000254\,\text{m}$.

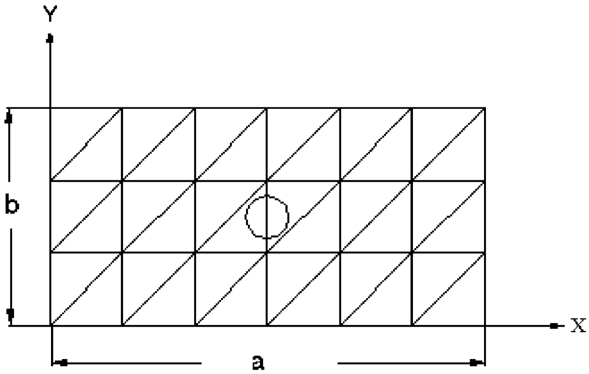

Fig. 3. Uniform mesh layout in a rectangular plate with a central circular hole.

4. Conclusions

The subparametric element developed in this paper can effectively handle free vibration problem with cutouts of arbitrary shapes without going for special types of meshing around the cutout boundary. Moreover the element does not get hindered while dealing with cutouts in very thin plates. Last but not the least accuracy can be achieved by using very less number of elements.

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